Calculation of the vertical velocity of geophysical flows with Mass Consistent Models

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A primary problem in geophysical fluid dynamics is the calculation of the vertical component of the velocity field $\mathbf{V}_{\rm T} = u_{\rm T}\mathbf{i} + v_{\rm T}\mathbf{j} + w_{\rm T}\mathbf{k}$ in a bounded region Ω with boundary Γ. Methods such as the integration of the equation $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$ and the so-called Omega equation, have been proposed [1]. In this work it is shown that Variational Mass Consistent Models (VMCM) [2−5] permit the exact calculation of the vertical component $w_{\rm T}$. The VMCM assume that an initial field $\mathbf{V}^0(x)$ is obtained by interpolation of discrete data. Then an adjusted field $\mathbf{V}$ is computed by means of the following conditions: (i) $\mathbf{V}$ minimizes the functional $J(\mathbf{V}) = \int_\Omega \left(\mathbf{V} - \mathbf{V}^0\right) \cdot \mathbb{S} \left(\mathbf{V} - \mathbf{V}^0\right) \, d\Omega$, where $\mathbb{S}(x)$ is a symmetric and positive definite matrix, (ii) $\mathbf{V}$ satisfies $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$ in Ω; if $\mathbf{n}$ denotes the outward unit normal to Γ and Γ_N is the part of Γ where an estimation of $\mathbf{V}_{\rm T} \cdot \mathbf{n}$ is known, (iii) $\mathbf{V}$ is subject to the boundary condition $\mathbf{V} \cdot \mathbf{n} = \mathbf{V}_{\rm T} \cdot \mathbf{n}$ on Γ_N.

A decomposition argument of the space $\mathbf{L}^2(\Omega)$ is used to prove the existence and uniqueness of $\mathbf{V}$ [4], which is given by $\mathbf{V} = \mathbf{V}^0 + \mathbb{S}^{-1}\nabla\lambda$ where λ satisfies the boundary condition $\lambda = 0$ on $\Gamma_D \equiv \Gamma \setminus \Gamma_N$. Thus $\mathbf{V}$ is obtained by solving the elliptic problem $L\lambda = \nabla \cdot \mathbf{V}^0$ in Ω, with $\mathcal{L}\lambda = \left(\mathbf{V}_{\rm T} - \mathbf{V}^0\right) \cdot \mathbf{n}$ on Γ_N, $\lambda = 0$ on Γ_D, where $L \equiv -\nabla \cdot \mathbb{S}^{-1}\nabla$, $\mathcal{L} \equiv \mathbf{n} \cdot \mathbb{S}^{-1}\nabla$. The standard problem solved in meteorology is $L\lambda = \nabla \cdot \mathbf{V}^0$ in Ω, $\mathcal{L}\lambda = 0$ on $\Gamma_{z=h(x,y)}$ =topography, $\lambda = 0$ on $\Gamma_D = \Gamma \setminus \Gamma_h$, with a diagonal and constant matrix $\mathbb{S} = \left\{\delta_{ij}\alpha_i^2\right\}$ and $\mathbf{V}^0 = u^0\mathbf{i} + v^0\mathbf{j}$.

To the date the following problems are open: (i) The field $\mathbf{V} = \mathbf{V}^0 + \mathbb{S}^{-1}\nabla\lambda$ depends critically of α_i but there is no consensus about the manner to compute α_i, (ii) there is no rigorous proof that the vertical component $w_{\rm T}$ can be computed with VMCM. In this work the following results are proved rigorously: (i) If the true horizontal field $\mathbf{V}_{\rm Th} = u_{\rm T}\mathbf{i} + v_{\rm T}\mathbf{j}$ is known, there is a unique vertical component $w_{\rm T}$ that satisfies $\boldsymbol{\nabla} \cdot \mathbf{V}_{\rm T} = 0$ and $\mathbf{V}_{\rm T} \cdot \mathbf{n} = 0$ on $\Gamma_{z=h(x,y)}$. Thus we get a unique field $\mathbf{V}_{\rm T} = u_{\rm T}\mathbf{i} + v_{\rm T}\mathbf{j} + w_{\rm T}\mathbf{k}$. (ii) If the initial field $\mathbf{V}^0$ is $\mathbf{V}_{\rm Th}$ and we solve the standard problem with $\Gamma_D = \Gamma_{z=h(x,y)}$, there is an optimal value $\alpha_{3,\rm opt}$ of α_3 for which the horizontal components of the adjusted field $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ coincide with those of $\mathbf{V}_{\rm T}$. The component w also coincides with $w_{\rm T}$ except in a vicinity of the lateral boundaries $\Gamma_{x=0,x_{\max}} \equiv \{\text{planes } x = 0,\ x = x_{\max}\}$, $\Gamma_{y=0,y_{\max}} \equiv \{\text{planes } y = 0,\ y = y_{\max}\}$ where w is discontinuous [5]. (iii) This last deficiency is eliminated with suitable boundary conditions. Thus we solve the problem of estimating α_3. The results are independent of α_1, α_2, this opens the possibility of computing a better field $\mathbf{V}$ by means of additional data (such as pressure data) or parameter estimation methods (such as genetic algorithms [3]) when $\mathbf{V}^0$ is an estimation of $\mathbf{V}_{\rm Th}$.

References.

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